SOME PROPERTIES OF MAPPINGS ON GENERALIZED TOPOLOGICAL SPACES

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Abstract. This paper considers generalizations of open mappings, closed mappings, pseudo-open mappings, and quotient mappings from topological spaces to generalized topological spaces. Characterizations of these classes of mappings are obtained and some relationships among these classes are established.

1. Introduction

Generalized topological spaces are an important generalization of topological spaces. Let X be a set and $\mu \subseteq \exp X$. Then μ is called a generalized topology on X and (X, μ) is called a generalized topological space, if μ satisfies the following two properties:

(1) $\emptyset \in \mu$,

(2) Any union of elements of μ belongs to μ.

There are plenty of generalized topological spaces which are not topological spaces. For instance, let $X = \{a, b, c\}$ and put $\mu = \{\emptyset, \{a, b\}, \{b, c\}, X\}$, then (X, μ) is a generalized topological space but it is not a topological space. Many interesting results and valuable applications of generalized topological spaces were previously considered (see [1, 2, 6, 7, 8, 10, 12, 13], for example).

Min investigated (μ, ν)-continuous mappings from a generalized topological space (X, μ) to another generalized topological space (Y, ν), which were generalized continuous mappings introduced by Csaszar in [1]. He then obtained some interesting characterizations for (μ, ν)-continuous mappings ([10]). In this paper, we make further investigations for (μ, ν)-continuous mappings and develop some aspect of mapping theory in generalized topology. We consider some mappings which are defined on topological spaces for example open mappings, closed mappings, pseudo-open mappings, and quotient mappings, and we generalize them to the setting of generalized topological spaces (see Definition 3). We obtain characterizations of these classes of mappings, and establish some relationships among these classes.

Throughout this paper, generalized topological spaces are denoted by GT-spaces and (μ, ν)-continuous mappings are in fact a generalization of continuous mappings and are denoted by g-continuous mappings (see Definition 1). (X, μ) and (Y, ν) always denote GT-spaces. Every mapping in this paper is assumed to be surjective, and every GT-space is assumed to be "strong". Here, a GT-space (X, μ) is said to be strong if $X \in \mu$ [2].

Let (X, μ) be a GT-space and $B \subseteq X$. B is called μ-open (resp. μ-closed) in X if $B \in \mu$ (resp. X-$B \in \mu$). $I(B)$ denotes the largest μ-open subset of B, i.e., $I(B)$ is the union of all μ-open subsets of B, and is called the interior of B in (X, μ). $C(B)$ denotes the smallest μ-closed subrset of B, i.e., $C(B)$ is the intersection of

all μ-closed subsets of B, and is called the closure of B in (X, μ). For $x \in X$, let U_x be a μ-open subset containing x, i.e., $x \in U_x \in \mu$. Then x is called a μ-cluster point of B in (X, μ) if $U_x \bigcap (B - \{x\}) \neq \emptyset$ for each U_x. Let $d(B)$ denote the set of all μ-cluster points of B in (X, μ). Thus, $x \in d(B)$ if and only if $x \in C(B - \{x\})$.

2. Preliminaries

The following notations are used throughout this paper. Let (X, μ) be a GT-space, let $x \in X$ and $\mathcal{F} \subseteq \exp X$.
(1) $\bigcup \mathcal{F} = \bigcup \{F : F \in \mathcal{F}\}$.
(2) $\bigcap \mathcal{F} = \bigcap \{F : F \in \mathcal{F}\}$.
(3) $\mu_x = \{U : x \in U \in \mu\}$.
(4) $N(x) = \bigcap \mu_x$.

In generalized topological spaces, some basic properties of interior subsets, closure subsets, μ-open subsets, μ-closed subsets, and μ-cluster points are given in the following proposition (see [1, 2, 10], for example).

Proposition 1. *Let B be a subset of a $GT-$space (X, μ). Then the following hold.*
(1) $I(B) \subseteq B \subseteq C(B)$.
(2) $I(I(B)) = I(B)$ and $C(C(B)) = C(B)$.
(3) If $B' \subseteq B$, then $I(B') \subseteq I(B)$, $C(B') \subseteq C(B)$ and $d(B') \subseteq d(B)$.
(4) $I(B) = B \Longleftrightarrow B$ is μ-open in $X \Longleftrightarrow$ for each $x \in B, U_x \subseteq B$ for some $U_x \in \mu_x$.
(5) $C(B) = B \Longleftrightarrow B$ is μ-closed in $X \Longleftrightarrow$ for each $x \in X - B, U_x \bigcap B = \emptyset$ for some $U_x \in \mu_x$.
(6) $C(B) = X - I(X - B)$ and $I(B) = X - C(X - B)$.
(7) $x \in C(B) \Longleftrightarrow U_x \bigcap B \neq \emptyset$ for each $U_x \in \mu_x$.
(8) $x \in I(B) \Longleftrightarrow U_x \subseteq B$ for some $U_x \in \mu_x$.
(9) $C(B) = B \bigcup d(B)$.
(10) $x \notin d(\{x\})$ for each $x \in X$.

Definition 1. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Then f is called g-continuous* [1] *if $f^{-1}(U) \in \mu$ for each $U \in \nu$.*

Notice that g-continuous mappings here are the same as (μ, ν)-continuous mappings. Thus, the characterizations of g-continuous mappings in the next proposition are the characterizations of (μ, ν)-continuous mappings obtained in [10].

Proposition 2. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Then the following are equivalent.*
(1) f is g-continuous.
(2) $f^{-1}(F)$ is μ-closed in X for each ν-closed subset F of Y.
(3) $f(C(A)) \subseteq C(f(A))$ for each subset A of X.
(4) $C(f^{-1}(B)) \subseteq f^{-1}(C(B))$ for each subset B of Y.
(5) $f^{-1}(I(B)) \subseteq I(f^{-1}(B))$ for each subset B of Y.
(6) For each $x \in X$, if $f(x) \in V \in \nu$, then $f(U) \subseteq V$ for some $U \in \mu_x$.

Given a subset in a generalized topological space, we can also induce a subspace of the generalized topological space. Furthermore, the relationship between a generalized topological space and its subspaces regarding closure and interior is discussed in the following proposition.

Proposition 3. *Let (X, μ) be a GT-space. If X' is a subset of X, A is a subset of X', and put $\mu' = \{U \bigcap X' : U \in \mu\}$. Then the following properties hold.*

(1) (X', μ') is a GT-space, which is called a subspace of (X, μ).

(2) $C_{X'}(A) = C(A) \bigcap X'$, where $C_{X'}(A)$ is the closure of A in (X', μ').

(3) $I(A) \subseteq I_{X'}(A) \bigcap I(X')$, where $I_{X'}(A)$ is the interior of A in (X', μ').

Proof. (1) It is clear.

(2) Let $x \in C_{X'}(A)$. For each $U \in \mu_x$, $U \bigcap X' \in \mu'_x$. By Proposition 1 (7), $(U \bigcap X') \bigcap A \neq \emptyset$, and so $U \bigcap A \neq \emptyset$. Thus $x \in C(A)$. Note that $x \in X'$. So $x \in C(A) \bigcap X'$. On the other hand, let $x \in C(A) \bigcap X'$, then $x \in X'$ and $x \in C(A)$. For each $U' \in \mu'_x$, there is $U \in \mu_x$ such that $U' = U \bigcap X'$. By Proposition 1 (7), $U \bigcap A \neq \emptyset$. Notice that $A \subseteq X'$. So $U' \bigcap A = (U \bigcap X') \bigcap A = U \bigcap A \neq \emptyset$. Thus $x \in C_{X'}(A)$.

(3) From Proposition 1 (3), $I(A) \subseteq I(X')$ because $A \subseteq X'$. So we only need to prove that $I(A) \subseteq I_{X'}(A)$. Let $x \in I(A)$, then there is $U \in \mu_x$ such that $x \in U \subseteq A \subseteq X'$. It suffices to prove that $x \in I_{X'}(A)$. In fact, since $U \bigcap X' = U$, $U \in \mu'_x$, and so $x \in I_{X'}(A)$. □

Remark 1. *(1) In Proposition 3 (3), "$\subseteq$" can not be replaced by "$=$". In fact, let $X = \{x, y, z\}$, $\mu = \{X, \{x, y\}, \{y, z\}, \phi\}$, $X' = \{x, y\}$ and $\mu' = \{X', \{y\}, \phi\}$. Then (X', μ') is a subspace of (X, μ). According to the definition, $I(\{y\}) =$ the union of all $\mu-$open subsets of $\{y\} = \emptyset$. Similar, $I_{X'}(\{y\}) = \{y\}$ and $I(X') = \{x, y\}$. So $I_{X'}(\{y\}) \bigcap I(X') = \{y\} \bigcap \{x, y\} = \{y\} \neq I(\{y\})$.*

(2) Given a subspace (X', μ') of a GT-space (X, μ), it is clear that B is a $\mu'-$closed subset of (X', μ') if and only if there is a $\mu-$closed subset F of (X, μ) such that $B = F \cap X'$.

Definition 2. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping and $X' \subseteq X$. Then $h : (X', \mu') \longrightarrow (f(X'), \nu')$ is called a restriction of f on (X', μ'), if $h(x) = f(x)$ for each $x \in X'$, where $\mu' = \{U \bigcap X' : U \in \mu\}$ and $\nu' = \{V \bigcap f(X') : V \in \nu\}$. As usual, a restriction of f on (X', μ') is denoted by $f|_{X'}$.*

Proposition 4. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping and $X' \subseteq X$. If f is g-continuous, then its restriction $f|_{X'}$ on (X', μ') is g-continuous.*

Proof. Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be g-continuous, $A \subseteq X'$, and the restriction (X', μ') be $f|_{X'} : (X', \mu') \longrightarrow (f(X'), \nu')$,where $\mu' = \{U \bigcap X' : U \in \mu\}$ and $\nu' = \{V \bigcap f(X') : V \in \nu\}$. By Propositions 3 (2) and 2 (3), so we have $h(C_{X'}(A)) = h(C(A) \bigcap X') = f(C(A) \bigcap X') \subseteq f(C(A)) \bigcap f(X') \subseteq C(f(A)) \bigcap f(X') = C_{f(X')}(f(A))$,where $C_{X'}(A)$ and $C_{f(X')}(f(A))$ are the closures of A in (X', μ') and $f(A)$ in $(f(X'), \nu')$ respectively. Thus $f|_{X'}$ is g-continuous by Proposition 2 (3). □

There are many useful mappings defined on topological spaces. Here we consider the classes of open mappings [3], closed mappings [3], pseudo-open mappings [4], and quotient mappings [3]. We generalize these classes of mappings on topological spaces to the classes of g-open mappings, g-closed mappings, g-pseudo-open mappings, and g-quotient mappings respectively, defined on generalized topological spaces by the following definition.

Definition 3. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Then*

(1) f is called a g-open mapping if $f(U) \in \nu$ for each $U \in \mu$.

(2) f is called a g-closed mapping if $f(F)$ is a ν-closed subset of Y for each μ-closed subset F of X.

(3) f is called a g-pseudo-open mapping if for each $y \in Y$ and $f^{-1}(y) \subseteq U \in \mu$, then $y \in I(f(U))$.

(4) f is called a g-quotient mapping if for each $V \subseteq Y$, $f^{-1}(V) \in \mu$ implies $V \in \nu$.

We note that the generalized quotient map of a generalized quotient topology ([13], Definition 3.1) is a g-quotient mapping according to our Definition 3 (4). But a g-quotient mapping need not be a generalized quotient map because that the corresponding generalized topology need not be a generalized quotient topology. Thus, g-quotient mappings here are generalizations of generalized quotient maps.

Furthermore, we generalize hereditary properties of mappings on topological spaces [3, 9] to the context of generalized topological spaces.

Definition 4. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Then f is called a hereditarily g-open mapping (resp. hereditarily g-closed mapping, hereditarily g-pseudo-open mapping, hereditarily g-quotient mapping) if $f|_{f^{-1}(Y')} : (f^{-1}(Y'), \mu') \longrightarrow (Y', \nu')$ is g-open (resp. g-closed, g-pseudo-open, g-quotient) for each $Y' \subseteq Y$. Where $\mu' = \{U \bigcap f^{-1}(Y') : U \in \mu\}$ and $\nu' = \{V \bigcap Y' : V \in \nu\}$.*

Proposition 5. *Every g-open mapping (resp. g-closed mapping, g-pseudo-open mapping) is hereditarily g-open (resp. g-closed, g-pseudo-open).*

Proof. Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. For each $Y' \subseteq Y$, put $X' = f^{-1}(Y')$, $\mu' = \{U \bigcap X' : U \in \mu\}, \nu' = \{V \bigcap Y' : V \in \nu\}$, and $h = f|_{X'} : (X', \mu') \longrightarrow (Y', \nu')$. Notice that all mappings in this paper are assumed to be surjective, hence $f(X') = Y'$.

(1) Assume that f is g-open. Let $W \in \mu'$, then there is $U \in \mu$ such that $W = U \bigcap X'$. So $h(W) = h(U \bigcap X') = f(U \bigcap f^{-1}(Y')) = f(U) \bigcap f(f^{-1}(Y'))) = f(U) \bigcap Y'$. Since f is g-open, $f(U) \in \nu$, so $h(W) \in \nu'$. This shows that h is g-open, so that f is hereditarily g-open.

(2) Assume that f is g-closed. Let F is a μ'-closed subset of (X', μ'). By Propositions 1 (5) and 3 (2), there is a μ-closed subset E of (X, μ) such that $F = E \bigcap X'$. So $h(F) = h(E \bigcap X') = f(E \bigcap f^{-1}(Y')) = f(E) \bigcap Y'$. Since f is g-closed, $f(E)$ is a ν-closed subset of (Y, ν), so $h(F)$ is a ν'-closed subset of (Y', ν'). This shows that h is g-closed, so that f is hereditarily g-closed.

(3) Assume that f is g-pseudo-open. Let $y \in Y'$ and $h^{-1}(y) \subseteq W \in \mu'$, then there is $U \in \mu$ such that $W = U \bigcap X'$. Since $h^{-1}(y) \subseteq W \subseteq X'$ and $h = f|_{X'}$, $f^{-1}(y) = h^{-1}(y)$. So $f^{-1}(y) \subseteq W \subseteq U$. Since f is g-pseudo-open, $y \in I(f(U))$, hence $y \in I(f(U)) \bigcap Y'$. Since $I(f(U)) \bigcap Y' \in \nu', I(f(U)) \bigcap Y' = I_{Y'}(I(f(U)) \bigcap Y')$. On the other hand, from Proposition 1 (3), we have $I_{Y'}(I(f(U)) \bigcap Y') \subseteq I_{Y'}(f(U) \bigcap Y') = I_{Y'}(f(U \bigcap X')) = I_{Y'}(h(U \bigcap X')) = I_{Y'}(h(W))$ which gives $y \in I_{Y'}(h(W))$. This proves that h is g-pseudo-open, so that f is hereditarily g-pseudo-open. □

Remark 2. *A g-quotient mapping need not be hereditarily g-quotient. In fact, we will show [Theorem 4] that a mapping is g-pseudo-open if and only if it is hereditarily g-quotient, but a g-quotient mapping need not be a g-pseudo-open mapping by Remark 3.*

3. The Main Results

In this section, we obtain characterizations of $g-$open mappings, $g-$closed mappings, $g-$pseudo-open mappings, and $g-$quotient mappings on generalized topological spaces.

Theorem 1. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Then the following conditions are equivalent.*

(1) f is a g-open mapping.

(2) If $B \subseteq Y$, then $f^{-1}(C(B)) \subseteq C(f^{-1}(B))$.

(3) If $A \subseteq X$, then $f(I(A)) \subseteq I(f(A))$.

(4) For each $x \in X$, if $x \in U \in \mu$, then $f(x) \in V \subseteq f(U)$ for some $V \in \nu$.

Proof. (1) $\Longrightarrow$ (2): Assume that f is a g-open mapping. Let $x \in f^{-1}(C(B))$, then $f(x) \in ff^{-1}(C(B)) = C(B)$. On the other hand, whenever $U_x \in \mu_x$ implies $f(x) \in f(U_x)$, then $f(U_x) \in \nu_{f(x)}$ because f is a g-open mapping. It follows from Proposition 1 (7) that $f(U_x) \bigcap B \neq \emptyset$. Choose $y \in f(U_x) \bigcap B$. Then there is $x' \in U_x$ such that $y = f(x') \in B$, i.e., $x' \in f^{-1}(B)$. Therefore $x' \in U \cap f^{-1}(B) \neq \emptyset$ which gives $x \in C(f^{-1}(B))$ by using Proposition 1 (7).

(2) $\Longrightarrow$ (3): Assume that the condition (2) holds. Let $A \subseteq X$. By Proposition 1 (1), $I(A) \subseteq A \subseteq f^{-1}f(A)$, so $I(A) \subseteq I(f^{-1}f(A))$. By Proposition 1 (6), $I(f^{-1}f(A)) = X - C(X - f^{-1}f(A)) = X - C(f^{-1}(Y - f(A)))$, so $I(A) \subseteq X - C(f^{-1}(Y - f(A)))$. Since the condition (2) holds, $f^{-1}(C(Y - f(A))) = C(f^{-1}(Y - f(A)))$, and hence $I(A) \subseteq X - f^{-1}(C(Y - f(A))) = f^{-1}(Y - C(Y - f(A))) = f^{-1}(I(f(A)))$. It follows that $f(I(A)) \subseteq ff^{-1}(I(f(A))) = I(f(A))$.

(3) $\Longrightarrow$ (1): Assume that the condition (3) holds. Let $U \in \mu$, then $I(U) = U$, and $f(U) = f(I(U) \subseteq I(f(U))$. On the other hand, $I(f(U)) \subseteq f(U)$ by using Proposition 1 (1). Thus, $I(f(U)) = f(U)$ which means $f(U) \in \nu$ by Proposition 1 (4). This proves that f is a g-open mapping.

(1) $\Longrightarrow$ (4): Assume that f is a g-open mapping. For each $x \in X$, if $x \in U \in \mu$, then $f(x) \in f(U) \in \nu$, for $V = f(U)$.

(4) $\Longrightarrow$ (1): Assume that the condition (4) holds. Let $U \in \mu$. For each $y \in f(U)$, there is $x_y \in U$ such that $y = f(x_y)$. So there is $V_y \in \nu$ such that $y \in V_y \subseteq f(U)$. It follows that $f(U) = \bigcup\{V_y : y \in f(U)\}$, and $\bigcup\{V_y : y \in f(U)\} \in \nu$ because Y is a GT. This proves that f is a g-open mapping. □

From Theorem 1 and Proposition 2 above, the next Corollary holds immediately.

Corollary 1. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a g-continuous mapping. Then the following conditions are equivalent.*

(1) f is a g-open mapping.

(2) If $B \subseteq Y$, then $f^{-1}(C(B)) = C(f^{-1}(B))$.

The following theorem characterizing g-closed mappings in generalized topological spaces is a generalization of the closed mapping theorem in topological spaces (see [3, Theorem 1.4.13] or [5, Lemma 2.3], for example)

Theorem 2. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Then the following conditions are equivalent.*

(1) f is a g-closed mapping.

(2) If $A \subseteq X$, then $C(f(A)) \subseteq f(C(A))$.

(3) If $B \subseteq Y$ and $U \in \mu$ such that $f^{-1}(B) \subseteq U$, then there is $V \in \nu$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

(4) If $y \in Y$ and $U \in \mu$ such that $f^{-1}(y) \subseteq U$, then there is $V \in \nu$ such that $y \in V$ and $f^{-1}(V) \subseteq U$.

Proof. (1) $\Longrightarrow$ (2): Assume that f is a g-closed mapping. Let $A \subseteq X$, then $C(A)$ is a μ-closed subset of X, and so $f(C(A))$ is a ν-closed subset of Y. Since $A \subseteq C(A)$, $f(A) \subseteq f(C(A))$, hence $C(f(A)) \subseteq f(C(A))$ by using Proposition 1 (3).

(2) $\Longrightarrow$ (1): Assume that the condition (2) holds. Let A be a μ-closed subset of X. Then $C(A) = A$, and $f(A) = f(C(A)) = C(f(A))$. By Proposition 1 (5), $f(A)$ is a ν-closed subset of Y. This proves that f is a g-closed mapping

(1) $\Longrightarrow$ (3): Assume that f is a g-closed mapping. Let $B \subseteq Y$ and $U \in \mu$ such that $f^{-1}(B) \subseteq U$. Put $V = Y - f(X - U)$. It suffices to check the following three claims.

Claim 1. $V \in \nu$.

Since $X - U$ is a μ-closed subset of X and f is g-closed, so $f(X - U)$ is a ν-closed subset of Y. It follows that $V = Y - f(X - U) \in \nu$.

Claim 2. $B \subseteq V$.

$f^{-1}(B) \subseteq U$ implies $X - U \subseteq X - f^{-1}(B)$, hence $f(X - U) \subseteq f(X - f^{-1}(B)) = f(f^{-1}(Y - B)) = Y - B$. It follows that $B \subseteq Y - f(X - U) = V$.

Claim 3. $f^{-1}(V) \subseteq U$.

$f^{-1}(V) = f^{-1}(Y - f(X - U)) = f^{-1}(Y) - f^{-1}(f(X - U)) \subseteq X - (X - U) = U$.

(3) $\Longrightarrow$ (4): It is clear.

(4) $\Longrightarrow$ (1): Assume that the condition (3) holds. Let F be a μ-closed subset of X. We only need to prove that $f(F)$ is a ν-closed subset of Y.

Let $y \in Y - f(F)$, then $f^{-1}(y) \bigcap F = \emptyset$, i.e., $f^{-1}(y) \subseteq X - F$. Notice that $X - F \in \mu$, then by (4) there is $V \in \nu$ such that $y \in V$ and $f^{-1}(V) \subseteq X - F$, i.e., $f^{-1}(V) \bigcap F = \emptyset$. It follows that $V \bigcap f(F) = ff^{-1}(V) \bigcap f(F) = f(f^{-1}(V) \bigcap F) = f(\emptyset) = \emptyset$. By Proposition 1 (5), $f(F)$ is a μ-closed subset of Y. □

The next Corollary holds immediately from Theorem 2 and Proposition 2 above.

Corollary 2. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Then the following conditions are equivalent.*

(1) f is a g-closed mapping.

(2) If $A \subseteq X$, then $C(f(A)) = f(C(A))$.

We now discuss the characterization of g-quotient mappings. We then show the characterization of g-pseudo-open mappings.

Theorem 3. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Then the following conditions are equivalent.*

(1) f is a g-quotient mapping.

(2) For each subset F of Y, if $f^{-1}(F)$ is a μ-closed subset of (X, μ), then F is a ν-closed subset of (Y, ν).

Proof. (1) $\Longrightarrow$ (2): Assume that f is a g-quotient mapping. Let $F \subseteq Y$ such that $f^{-1}(F)$ is a μ-closed subset of (X, μ). Then $f^{-1}(Y - F) = X - f^{-1}(F) \in \mu$. Since f is a g-quotient mapping, $Y - F \in \nu$, and so F is a ν-closed subset of (Y, ν).

(2) $\Longrightarrow$ (1). It can be proved by the same method. □

Theorem 4. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Then the following conditions are equivalent.*

(1) f is a g-pseudo-open mapping.

(2) f is a hereditarily g-quotient mapping.
(3) If $B \subseteq Y$, then $C(B) \subseteq f(C(f^{-1}(B)))$.

Proof. (1) $\Longrightarrow$ (2): Assume that f is a g-pseudo-open mapping. Proposition 5 states that every g-pseudo-open mapping is hereditarily g-pseudo-open. It follows that $f|_{f^{-1}(Y')}$ is g-pseudo-open for each $Y' \subseteq Y$. Thus, it suffices to prove that every g-pseudo-open mapping f is a g-quotient mapping so that $f|_{f^{-1}(Y')}$ is g-quotient for each $Y' \subseteq Y$.

Let $V \subseteq Y$ such that $f^{-1}(V) \in \mu$. If $y \in V$, then $f^{-1}(y) \subseteq f^{-1}(V) \in \mu$. Notice that f is g-pseudo-open, hence $y \in I(ff^{-1}(V)) = I(V)$ for each $y \in V$. It follows that $V \in \nu$. This proves that f is a g-quotient mapping.

(2) $\Longrightarrow$ (3): Assume that f is a hereditarily g-quotient mapping. Let $B \subseteq Y$ and let $y \in C(B) = B \cup (C(B) - B)$.

(i) If $y \in B$, then $y \in B = f(f^{-1}(B)) \subseteq f(C(f^{-1}(B)))$.

(ii) If $y \in C(B) - B$, let $Y' = B \bigcup \{y\}$ and $\nu' = \{V \bigcap Y' : V \in \nu\}$, and let $X' = f^{-1}(Y') = f^{-1}(B) \bigcup f^{-1}(y)$, $\mu' = \{U \bigcap X' : U \in \mu\}$ and $h = f|_{X'} : (X', \mu') \longrightarrow (Y', \nu')$, then h is a g-quotient mapping because f is hereditarily g-quotient. Since $C(B)$ is the smallest $\mu-$closed subset of (X, μ) containing B and $C(B) \cap Y' = B \bigcup \{y\}$, B is not a g-closed subset of (Y', ν'), it follows from Theorem 3 that $h^{-1}(B)$ is not a g-closed subset of (X', μ'), and hence there is $x \in C_{X'}(h^{-1}(B)) - h^{-1}(B)$. Notice that $h^{-1}(B) = f^{-1}(B)$, so $C_{X'}(h^{-1}(B)) = C_{X'}(f^{-1}(B)) \subseteq C(f^{-1}(B))$ which gives $x \in C(f^{-1}(B))$, and hence $f(x) \in f(C(f^{-1}(B)))$. On the other hand, since $C_{X'}(h^{-1}(B)) \subseteq X'$ and $h^{-1}(B) = f^{-1}(B), x \in C_{X'}(h^{-1}(B)) - h^{-1}(B) \subseteq X' - f^{-1}(B) = f^{-1}(B) \bigcup f^{-1}(y) - f^{-1}(B) = f^{-1}(y)$, and hence $y = f(x) \in f(C(f^{-1}(B)))$. Finally, (i) and (ii) imply $C(B) \subseteq f(C(f^{-1}(B)))$.

(3) $\Longrightarrow$ (1): Assume that the condition (3) holds. Let $y \in Y$ and $f^{-1}(y) \subseteq U \in \mu$. Then $f^{-1}(y) \bigcap (X - U) = \emptyset$, so $y \notin f(X - U)$, i.e., $y \in Y - f(X - U)$. Since the condition (3) holds, $C(Y - f(U)) = f(C(f^{-1}(Y - f(U)))) = f(C(f^{-1}(Y) - f^{-1}f(U))) \subseteq f(C(X - U)) = f(X - U)$. By Proposition 1 (6), $I(f(U)) = Y - C(Y - f(U)) \supset Y - f(X - U)$. So $y \in Y - f(X - U)$ implies $y \in I(f(U))$. This proves that f is a g-pseudo-open mapping. □

It is not difficult to see that the Corollary below holds from Theorem 4 and Proposition 2. above.

Corollary 3. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Then the following conditions are equivalent.*
(1) f is a g-pseudo-open mapping.
(2) If $B \subseteq Y$, then $C(B) = f(C(f^{-1}(B)))$.

Finally, we can establish relationships among $g-$open mappings, $g-$closed mappings, $g-$pseudo-open mappings, and $g-$quotient mappings on generalized topological spaces.

Theorem 5. *Let $f : (X, \mu) \longrightarrow (Y, \nu)$ be a mapping. Consider the following conditions.*
(1) f is a g-open mapping.
(2) f is a g-closed mapping.
(3) f is a g-pseudo-open mapping.
(4) f is a g-quotient mapping.
Then (1) $\Longrightarrow$ (3) $\Longrightarrow$ (4) and (2) $\Longrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (3): Assume that f is a g-open mapping. Let $y \in Y$ such that $f^{-1}(y) \subseteq U \in \mu$. Then $y \in f(U)$ and $U = I(U)$. By Theorem 1, $y \in f(U) = f(I(U)) \subseteq I(f(U))$. This proves that f is a g-pseudo-open mapping.

(3) $\Longrightarrow$ (4): It holds from Theorem 2. Here, we give a proof by using characterizations of g-pseudo-open mappings and g-quotient mappings. Assume that f is a g-pseudo-open mapping. Let $F \subseteq Y$ such that $f^{-1}(F)$ is a μ-closed subset of X. Then $C(f^{-1}(F)) = f^{-1}(F)$. By Theorem 2, $C(F) = f(C(f^{-1}(F))) = f(f^{-1}(F)) = F$. So F is a μ-closed subset of X. This proves that f is a g-quotient mapping.

(2) $\Longrightarrow$ (3): Assume that f is a g-closed mapping. Let $y \in Y$ such that $f^{-1}(y) \subseteq U \in \mu$. By Theorem 2, there is $V \in \mu$ such that $f^{-1}(y) \subseteq V \subseteq U$ and $f(V) \in \nu$. So $y \in f(V) = I(f(V)) \subseteq I(f(U))$. This proves that f is a g-pseudo-open mapping. $\square$

Remark 3. *Among g-open mappings, g-closed mappings, g-pseudo-open mappings and g-quotient mappings, the only implications that hold are those that can be obtained from Theorem 5. In fact, if both (X, μ) and (Y, ν) in Theorem 5 are topological spaces, then f is g-open (resp. g-closed, g-pseudo-open, g-quotient) if and only if f is open (resp. closed, pseudo-open, quotient). Moreover, among open mappings, closed mappings, pseudo-open mappings and quotient mappings, all other implications rather than those that are similar to Theorem 5 cannot hold. (see* [3, 9]*, for example).*

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